On the recurrence coefficients of the generalized little q-Laguerre polynomials

Galina Filipuk* and Christophe Smet†

Abstract

In this paper we consider a semi-classical variation of the weight related to the little q-Laguerre polynomials and obtain a second order second degree discrete equation for the recurrence coefficients in the three-term recurrence relation.

Key words: orthogonal polynomials, recurrence coefficients, discrete equations.
MSC: 33C47

1 Introduction

1.1 Orthogonal polynomials

Orthogonal polynomials appear in many areas of modern mathematics and mathematical physics [4, 10] (e.g., approximation theory, stochastic processes, random matrix theory and others). In this paper we are interested in discrete q-orthogonal polynomials on an exponential lattice. The orthogonality condition for discrete q-orthonormal polynomials is given by

$$\int_a^b p_k(x)p_n(x)w(x)d_qx = \delta_{k,n},$$

where the q-integral [8] is defined by

$$\int_a^b f(x)d_qx = b(1-q)\sum_{n=0}^{\infty} q^n f(bq^n) - a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n).$$

Here the weight function w is supported on the exponential lattice

$$\{aq^n,\ bq^n \,|\, n \in \mathbb{N}_0\}$$

and $\delta_{k,n}$ is the Kronecker delta.

The classical examples include little q-Laguerre polynomials, which are orthogonal on the exponential lattice $\{q^k \,|\, k \in \mathbb{N}_0\}$ with respect to the weight function

$$w(x) = x^\alpha (qx;q)_\infty, \quad \alpha > -1, \quad q \in (0,1),$$

*Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, Warsaw 02-097, Poland. Email: filipuk@mimuw.edu.pl

†Department of Mathematics, KU Leuven, Celestijnenlaan 200 B box 2400, BE-3001, Leuven, Belgium. Email: chr.smet@gmail.com

where

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

They can be written in terms of the basic hypergeometric function ${}_2\phi_1$.

One of the main features of orthogonal polynomials is the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x).$$

Here $\{p_n(x)\}$ are orthonormal polynomials of degree n and the coefficients a_n and b_n are usually referred to as the recurrence coefficients. They possess a number of important properties. For instance, they can be expressed in terms of determinants containing the moments of the orthogonality measure [4]. Moreover, for classical orthogonal polynomials they are known explicitly. For other, non-classical, polynomials the recurrence coefficients are not known explicitly and sometimes they can be expressed in terms of the solutions of discrete (including q-discrete) or continuous Painlevé equations. The Painlevé equations in their turn have many remarkable applications in modern mathematics and mathematical physics (see for instance [5] and the references therein). There are a few examples of relations of q-orthogonal polynomials on an exponential lattice to the q-discrete Painlevé equations: the weight

$$w(x) = (q^4x^4;q^4)_\infty$$

on $\{\pm q^k \mid k \in \mathbb{N}_0\}$ and $q\mathrm{P_I}$ [16]; the weight

$$w(x) = |x|^\alpha (q^2x^2;q^2)_\infty (cq^2x^2;q^2)_\infty$$

on $\{\pm q^k \mid k \in \mathbb{N}_0\}$ and $\alpha q\mathrm{P_V}$ [1, 16]; the weight

$$w(x) = x^\alpha (q^2x^2;q^2)_\infty$$

on $\{q^k \mid k \in \mathbb{N}_0\}$ and $q\mathrm{P_V}$ [1]. For other examples of relations of the recurrence coefficients for the orthogonal polynomials, not necessarily supported on an exponential lattice, see for instance [2, 3, 6, 7, 13, 14] and the references therein. One of the methods to derive the nonlinear discrete equations for the recurrence coefficients is by using the ladder operators.

1.2 Ladder operators

In the case of discrete q-orthogonal polynomials on the exponential lattice, the ladder operators were first considered in [11]. We repeat the main statements which we use later on to be self-contained following [11] and [1, Section 1.3].

The q-difference operator is given by

$$(D_qf)(x) = \begin{cases} \frac{f(x)-f(qx)}{x(1-q)}, & x \neq 0, \\ f'(0) & x = 0. \end{cases}$$

Consider a weight function w on the exponential lattice $\{aq^n,\ bq^n \mid n \in \mathbb{N}_0\}$, such that $w(a/q) = w(b/q) = 0$ and the sequence of orthonormal polynomials $\{p_n\}$ of degree n with respect to this weight. Ismail [11] shows that the polynomials satisfy the following relation:

$$D_qp_n(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x)$$

with

$$A_n(x) = a_n \int_a^b \frac{u(qx) - u(y)}{qx - y} p_n(y) p_n(y/q) w(y) d_q y, \tag{1}$$

$$B_n(x) = a_n \int_a^b \frac{u(qx) - u(y)}{qx - y} p_n(y) p_{n-1}(y/q) w(y) d_q y. \tag{2}$$

Here the function u, called the potential, is defined by the following formula:

$$-u(qx) w(qx) = D_q w(x). \tag{3}$$

Furthermore, the following relations (compatibility conditions) hold:

$$B_n + B_{n+1} = (x - b_n) \frac{A_n}{a_n} + (q-1) x \sum_{j=0}^{n} \frac{A_j}{a_j} - u(qx), \tag{4}$$

$$a_{n+1} A_{n+1} - a_n^2 \frac{A_{n-1}}{a_{n-1}} = (x - b_n) B_{n+1} - (qx - b_n) B_n + 1. \tag{5}$$

Relations (4), (5) are important in deriving nonlinear discrete equations for the recurrence coefficients, which in some cases can be further reduced to (q-)discrete Painlevé equations.

2 Main results

In this paper we study the recurrence coefficients for the weight functions supported on the exponential lattice $\{q^k \mid k \in \mathbb{N}_0\}$ and satisfying the q-difference equation (3) with

$$u(x) = \frac{k_1 q}{1-q} \frac{1}{x} + \frac{k_2 x + k_3}{1-q}, \quad k_1 \neq 0, \quad k_2 \neq 0, \tag{6}$$

and conditions $w(0) = w(1/q) = 0$. At the end of this section we discuss the existence of such weight functions and consider a few instructive examples. In the following we assume that the sequence of polynomials $\{p_n\}$ is orthonormal with respect to the weight function with potential (6) and hence the orthogonality relation takes the form

$$\int_0^1 p_m(x) p_n(x) w(x) d_q x = \delta_{m,n}.$$

It is straightforward to calculate that

$$\frac{u(qx) - u(y)}{qx - y} = \frac{k_1}{(q-1)xy} + \frac{k_2}{1-q}.$$

Hence, the expressions (1) and (2) can be computed as follows:

$$A_n(x) = \frac{a_n R_n}{x(1-q)} + \frac{a_n k_2 q^{-n}}{1-q},$$

$$B_n(x) = \frac{r_n}{(1-q)x},$$

where

$$R_n = -k_1 \int_0^1 p_n(y) p_n(y/q) \frac{w(y)}{y} d_q y, \quad r_n = -a_n k_1 \int_0^1 p_n(y) p_{n-1}(y/q) \frac{w(y)}{y} d_q y$$

and we have used orthogonality in computing

$$\int_0^1 p_n(y) p_n(y/q) w(y) d_q y = q^{-n}, \quad \int_0^1 p_n(y) p_{n-1}(y/q) w(y) d_q y = 0.$$

The compatibility conditions (4), (5) give rise to the following system (after comparing the coefficients at the powers of x):

$$r_{n+1} + r_n = -b_n R_n - k_1, \tag{7}$$

$$R_n - k_2 q^{-n} b_n - k_3 - (1-q) \sum_{j=0}^{n} R_j = 0, \tag{8}$$

$$a_{n+1}^2 R_{n+1} - a_n^2 R_{n-1} = -b_n (r_{n+1} - r_n), \tag{9}$$

$$k_2 a_{n+1}^2 - k_2 q^2 a_n^2 = q^{1+n} - q^{2+n} - q^{2+n} r_n + q^{1+n} r_{n+1}. \tag{10}$$

We will use these equations to find expressions for the recurrence coefficients a_n, b_n of the sequence of orthonormal polynomials $\{p_n\}$ with respect to w. Multiplying (10) by q^{-2n-2} and taking a telescopic sum with $a_0 = r_0 = 0$, we get

$$a_n^2 = \frac{q^n (1 - q^n + r_n)}{k_2}. \tag{11}$$

Multiplying (9) by R_n, substituting $-b_n R_n$ from (7) and taking a telescopic sum, we obtain

$$a_n^2 R_n R_{n-1} = r_n (k_1 + r_n). \tag{12}$$

On the other hand, if we substitute the expression for b_n from (8) and the expression for a_n^2 from (11) into (9) and we collect the terms in r_n and r_{n+1}, we obtain

$$r_{n+1} \left(q R_{n+1} + R_n - k_3 - (1-q) \sum_{j=0}^{n} R_j \right) - r_n \left(q R_n + R_{n-1} - k_3 - (1-q) \sum_{j=0}^{n-1} R_j \right) \\ = (1 - q^n) R_{n-1} - q \left(1 - q^{n+1}\right) R_{n+1}.$$

The left hand side of this expression can easily be summed telescopically and in the resulting equation we recognize the expression for b_n from (8), so we get

$$r_n \left(R_{n-1} + k_2 b_n q^{-n} \right) = q^{n+1} R_n + q^n R_{n-1} - R_{n-1} - k_2 b_n q^{-n} - k_3.$$

If we multiply this by R_n, we can use expressions (7) to substitute $b_n R_n$ and (11), (12) to substitute $R_n R_{n-1}$. Eventually we get a quadratic equation for R_n:

$$R_n^2 - k_3 q^{-n-1} R_n = -k_2 q^{-2n-1} \left((1 + r_n)(1 + r_{n+1}) - (1 - k_1) \right). \tag{13}$$

From (11) and (12) we get

$$q^n (1 - q^n + r_n) R_n R_{n-1} = k_2 r_n (r_n + k_1). \tag{14}$$

We can use equations (13) and (14) to get a second order second degree difference equation for r_n. In order to derive such an equation (which we omit here explicitly as it is long and cumbersome) we can first eliminate R_n between equations (13) and (14) and then eliminate R_{n-1} between the obtained equation and (13) with n replaced by $n-1$.

Alternatively, we can first replace equation (8) by subtracting from it (8) with $n-1$ to eliminate the sum of R_n. We have the following equation instead of (8):

$$k_2 q b_{n-1} - k_2 b_n + q^n (q R_n - R_{n-1}) = 0. \tag{15}$$

From (7) we can find b_n in terms of r_n and R_n and substitute this expression into equations (8)–(10). Further, we use (11). Substituting (11) into (12), we can find R_{n-1} and, by taking (11) and (12) with n replaced by $n+1$, we can find R_{n+1} in terms of R_n and r_n (r_{n+1} respectively). We have

$$R_{n-1} = -\frac{k_2 q^{-n} r_n (r_n + k_1)}{(q^n - 1 - r_n) R_n}, \tag{16}$$

$$R_{n+1} = -\frac{k_2 q^{-1-n} r_{n+1} (r_{n+1} + k_1)}{(q^{n+1} - 1 - r_{n+1}) R_n}. \tag{17}$$

Substituting these expressions into (15), we can find R_n^2:

$$R_n^2 = -\frac{k_2 q^{-1-n} r_n (r_n + k_1)(r_n (q^n - 1 - r_{n+1}) + (q^n - 1)(r_{n+1} + k_1))}{(q^n - 1 - r_n)(r_{n-1}(q^n - 1 - r_n) + (q^n - 1)(r_n + k_1))}. \tag{18}$$

From (11) we have an expression of a_n^2 in terms of r_n. Using (18) and (13) we can get an expression of R_n in terms of r_{n-1}, r_n and r_{n+1}. Hence, using (11) we can get an expression of b_n in terms of r_{n-1}, r_n and r_{n+1}. Thus, we have proved the following theorem.

Theorem 2.1. *The recurrence coefficients a_n, b_n appearing in the three-term recurrence relation for the weight supported on the exponential lattice $\{q^k \mid k \in \mathbb{N}_0\}$ with the potential satisfying (6) and conditions $w(0) = w(1/q) = 0$ can be expressed in terms of r_n (by using (11), (7) with (18)), which is a solution of a second order second degree discrete equation (with respect to n) given by*

$$f^2 - 2fg + g^2 - k_3^2 q^{-2n-2} f = 0, \tag{19}$$

where f denotes the right hand side of (18) and g is the right hand side of (13).

Note that the discrete equation in the theorem factorizes if and only if $k_3 = 0$ and this allows us to express r_{n+1} in terms of r_{n-1} and r_n as a rational function. The connection is given by a q-Painlevé equation:

Theorem 2.2. *If $k_3 = 0$ in (6) then the variable $x_n = (1 + r_n)(1 - k_1)^{-1/2}$ satisfies qP_V [9] given by*

$$(x_n x_{n-1} - 1)(x_n x_{n+1} - 1) = \frac{\gamma \delta q^{2n} (x_n - \alpha)(x_n - 1/\alpha)(x_n - \beta)(x_n - 1/\beta)}{(x_n - \gamma q^n)(x_n - \delta q^n)}, \tag{20}$$

with

$$\alpha = \beta = \gamma = \delta = \frac{1}{p}$$

where $p=\sqrt{1-k_1}$. The initial conditions are given by

$$x_0=\frac{1}{p}\qquad\text{and}\qquad x_1=p-\frac{k_2}{qp}\left(\frac{\mu_1}{\mu_0}\right)^2.$$

x_n is related to the recurrence coefficients a_n and b_n of the orthogonal polynomials by

$$a_n^2=\frac{q^n}{k_2}\left(px_n-q^n\right) \tag{21}$$

and

$$b_n^2=-\frac{q^{2n+1}(px_n+px_{n+1}-1-p^2)^2}{k_2p^2(x_nx_{n+1}-1)}. \tag{22}$$

Proof. If $k_3=0$ then (19) simplifies to $f-g=0$. Putting $y_n=1+r_n$ and isolating the term containing y_ny_{n+1}, we find

$$\begin{aligned} 0 \;=\;& y_ny_{n+1}(y_n-q^n)^2(y_ny_{n-1}+k_1-1)=(k_1-1)(y_n-q^n)^2y_ny_{n-1}\\ &+(q^n-y_n)(k_1-1)y_n\left[(y_n-q^n)+(q^n-1)(y_n-1+k_1)\right]\\ &-q^n(y_n-1)(y_n-1+k_1)y_n\left[q^n(y_n-1)+(q^n-1)(k_1-1)\right]. \end{aligned}$$

With the substitution $y_n=\sqrt{1-k_1}x_n=px_n$ the first two terms can be completed to contain $(x_nx_{n-1}-1)(x_nx_{n+1}-1)$ and the equation simplifies to (20).

Initial conditions for a_n and b_n (and, hence, x_n) can be computed by using the fact that the recurrence coefficients can be expressed in terms of the Hankel determinants containing the moments of the orthogonality measure [4, Th. 4.2, p. 19; Ex. 3.1, p. 17]. In particular, we will need that $b_0=\mu_1/\mu_0$ where μ_k is the k'th moment of the weight w. Since $a_0=r_0=0$, we immediately find that $y_0=1$ and hence $x_0=1/p$. As for x_1, we know from (7) that $r_1=-b_0R_0-k_1$. Here, R_0 can be obtained from (8) with $n=0$ and $k_3=0$: we find that $R_0=q^{-1}k_2b_0$. This leads to

$$r_1=-k_1-\left(\frac{\mu_1}{\mu_0}\right)^2\frac{k_2}{q}$$

and hence

$$x_1=p-\frac{k_2}{pq}\left(\frac{\mu_1}{\mu_0}\right)^2.$$

The connection (21) between x_n and a_n follows immediately from (11). To obtain (22), we use the squared (7) to write

$$b_n^2=\frac{(r_{n+1}+r_n+k_1)^2}{R_n^2},$$

where R_n^2 can be substituted using (18). Finally, substitute r by x using $x_n=(1+r_n)/p$ and eliminate x_{n-1} using the q-Painlevé equation (20).

□

Clearly (in the case $k_3\neq 0$), we can also derive a third order difference equation for r_n from the system (7)–(10) as follows. We can get R_{n+1}^2 by either using relation (18) with

$n+1$ or by squaring (17) and using (18). Hence, we can set them equal and in the result, we get a (cumbersome) expression involving only r_{n-1}, r_n, r_{n+1}, r_{n+2}.

To find out which weights can give rise to a potential of the form (6), we notice that it is sufficient if

$$\frac{w(x/q)}{w(x)} = Ax^2 + Bx + C$$

for certain constants A, B, C, since an easy calculation shows that in that case the potential is given by (6) with $k_1 = 1 - C$, $k_2 = -Aq$ and $k_3 = -Bq$.

If we define

$$\begin{aligned} v_1^\alpha(x) = x^\alpha, \quad v_2^c(x) &= (cx;q)_\infty, \quad v_3^c(x) = (cx^2;q^2)_\infty, \\ v_4^c(x) &= (c/x;q)_\infty, \quad v_5^c(x) = (c/x^2;q^2)_\infty, \end{aligned}$$

then

$$\begin{aligned} \frac{v_1^\alpha(x/q)}{v_1^\alpha(x)} = q^{-\alpha}, \quad \frac{v_2^c(x/q)}{v_2^c(x)} &= \frac{q-cx}{q}, \quad \frac{v_3^c(x/q)}{v_3^c(x)} = \frac{q^2-cx^2}{q^2}, \\ \frac{v_4^c(x/q)}{v_4^c(x)} &= \frac{x}{x-c}, \quad \frac{v_5^c(x/q)}{v_5^c(x)} = \frac{x^2}{x^2-c}. \end{aligned}$$

Hence it is clear which products of v_i lead to a weight for which the potential satisfies (6). These include, among others, the little q-Laguerre weight, the weight in [1, Sect. 7.3], products of rational functions and the weights above, the weights in the following examples and others.

Example 1. In this example we consider the semi-classical little q-Laguerre weight

$$w(x) = x^\alpha (qx;q)_\infty (cqx;q)_\infty, \quad \alpha > 0, \tag{23}$$

on the positive exponential lattice $\{q^n \mid n \in \mathbb{N}_0\}$. The case $c = -1$ was considered in [1, Sect. 7.3]. The case $c = 0$ gives the little q-Laguerre weight (and, hence, the recurrence coefficients are known explicitly). We observe that $w(0) = w(1/q) = 0$.

The potential (3) is given by

$$u(x) = \frac{1}{1-q}\left(\frac{q}{x} - \frac{q^{1-\alpha}}{x} + q^{1-\alpha}(1+c) - cq^{1-\alpha}x\right)$$

and, hence, $k_1 = 1 - q^{-\alpha}$, $k_2 = -cq^{1-\alpha}$, $k_3 = (1+c)q^{1-\alpha}$ in (6). We assume that $c \neq 0$.

Since $k_3 = 0$ if and only if $c = -1$, we get that in this case the variable $x_n = q^{\alpha/2}(r_n+1)$ satisfies

$$(x_n x_{n-1} - 1)(x_n x_{n+1} - 1) = \frac{q^{2n+\alpha}(x_n - q^{\alpha/2})^2 (x_n - q^{-\alpha/2})^2}{(x_n - q^{n+\alpha/2})^2}, \tag{24}$$

which is a particular case of qP_V (20) (with $\alpha = \beta = \gamma = \delta = q^{\alpha/2}$). This coincides with the result in [1, Sect. 7.3]. Note that equation (24) for the variable r_n is given by

$$r_n^2(r_n+1-q^{-\alpha})^2 = q^{-2n}(r_n+1-q^n)^2((r_n+1)(r_{n+1}+1)-q^{-\alpha})((r_n+1)(r_{n-1}+1)-q^{-\alpha}). \tag{25}$$

Next we study initial conditions for the recurrence coefficients of the weight (23) for general c. Using (11), we have

$$a_n^2 = \frac{1}{c}\, q^{n-1+\alpha/2}\left(q^{n+\alpha/2} - x_n\right).$$

To find an expression of b_n in terms of x_n we square the expression of b_n from (7) and substitute (18). We can also use (25) to get rid of x_{n-1}. We get for $n \geq 1$

$$cb_n^2(x_n x_{n+1} - 1) = q^{2n}\left(1 + q^\alpha - q^{\alpha/2}(x_n + x_{n+1})\right)^2.$$

We also get that

$$b_0^2 = \frac{1}{c}\left(q^{\alpha/2}x_1 - 1\right).$$

Recalling that $b_0 = \mu_1/\mu_0$, we immediately get that the initial values are given by $x_0 = q^{\alpha/2}$ (since $r_0 = 0$) and

$$x_1 = q^{-\alpha/2}\left(1 + c\frac{\mu_1^2}{\mu_0^2}\right), \tag{26}$$

where μ_k is the k-th moment of the weight (23). In fact, we can also calculate μ_k by definition and get

$$\mu_k = (1-q)(q;q)_\infty(cq;q)_\infty\, {}_2\phi_1(0,0;cq;q;q^{\alpha+k+1}),$$

where the basic hypergeometric function ${}_2\phi_1$ is given by [12, Sect. 0.2, 0.4]

$${}_2\phi_1(a_1,a_2;b_1;q;z) = \sum_{\ell=0}^{\infty}\frac{(a_1;q)_\ell(a_2;q)_\ell}{(b_1;q)_\ell}\frac{z^\ell}{(q;q)_\ell}.$$

Note that for $c = q^\nu$ the last expression (up to a factor) can be written in terms of the modified q-Bessel function [15]

$$I_\nu^{(1)}(z,q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(z/2)^\nu\, {}_2\phi_1(0,0;q^{\nu+1};q;z^2/4)$$

with $z = 2q^{(\alpha+k+1)/2}$. Also note that a limiting case of Heine's transformation formula [12, formula (0.6.9)] allows us to write this q-hypergeometric function as a ${}_0\phi_1$-function.

Example 2. In this example we consider another semi-classical generalization of the little q-Laguerre weight:

$$w(x) = x^\alpha\frac{(qx;q)_\infty\left(\frac{c_1}{x};q\right)_\infty\left(\frac{qx}{c_1};q\right)_\infty}{\left(\frac{c_2}{x};q\right)_\infty},\quad \alpha > 0,\ c_1 < 0,\ c_2 < 0 \tag{27}$$

on the positive exponential lattice $\{q^n \mid n \in \mathbb{N}_0\}$. The case where $c_1 = c_2 = 1/c$ gives the weight from the previous example. Again, it is clear that $w(0) = w(1/q) = 0$. It is easy to calculate that for this weight we get

$$k_1 = 1 - \frac{c_2}{c_1}q^{-\alpha},\qquad k_2 = -\frac{q^{1-\alpha}}{c_1}\qquad\text{and}\qquad k_3 = \frac{c_2+1}{c_1}q^{1-\alpha}.$$

As mentioned earlier, to obtain a Painlevé equation, we need that $k_3 = 0$, hence $c_2 = -1$. So, following the outline given before, we see that

$$x_n = \sqrt{-c_1}\,q^{\alpha/2}(r_n+1)\qquad\text{and}\qquad a_n^2 = -c_1 q^{n+\alpha-1}(1 - q^n + r_n)$$

where x_n satisfies (20) with

$$\alpha = \beta = \gamma = \delta = \sqrt{-c_1}q^{\frac{\alpha}{2}}.$$

As for the initial conditions, we find that

$$x_0 = \sqrt{-c_1}q^{\frac{\alpha}{2}}$$

and

$$x_1 = \sqrt{-c_1}q^{\frac{\alpha}{2}}\left(-1 + \frac{c_2}{c_1}q^{-\alpha} + \frac{b_0^2}{c_1 q^{\alpha}}\right) + 1.$$

Moreover, $b_0 = \mu_1/\mu_0$, and it is easily seen that the k'th moment of this weight is given by

$$\mu_k = (1-q)(q;q)_\infty \left(\frac{q}{c_1};q\right)_\infty \frac{(c_1;q)_\infty}{(c_2;q)_\infty} {}_2\phi_1\left(0,0;\frac{q}{c_2};q;\frac{c_1}{c_2}q^{\alpha+1+k}\right).$$

Hence b_0 can be written, up to a factor, as a fraction of two modified q-Bessel functions with $q^{-\nu} = c_2$. Again, the q-hypergeometric function can alternatively be written as a ${}_0\phi_1$-function, using [12, formula (0.6.9)].

3 Discussion

In this paper we have shown that it is possible to study simultaneously recurrence coefficients in the three-term recurrence relation for a large class of weights by using the technique of ladder operators. The crucial point is to consider the potential (6) with parameters. This allows us to obtain a second degree second order discrete equation, which in some particular cases, can be further reduced to the discrete Painlevé equation. It is an interesting open problem to try to classify the weights which lead to the appearance of the discrete Painlevé equations for the recurrence coefficients.

Acknowledgements

GF is supported by MNiSzW Iuventus Plus Grant Nr 0124/IP3/2011/71. Part of this research was done when CS was visiting the University of Warsaw, and he is grateful for their hospitality.

References

[1] L. Boelen, *Discrete Painlevé Equations and Orthogonal Polynomials*, Ph.D. diss., Department of Mathematics K.U.Leuven, Leuven, Belgium, 2010.

[2] L. Boelen, G. Filipuk, and W. Van Assche, *Recurrence coefficients of generalized Meixner polynomials and Painlevé equations*, J. Phys. A: Math. Theor. 44 (2011), 035202 (19p).

[3] L. Boelen and W. Van Assche, *Discrete Painlevé equations for recurrence coefficients of semiclassical Laguerre polynomials*, Proc. Amer. Math. Soc. 138 (2010), pp. 1317–1331.

[4] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[5] P. A. Clarkson, *Painlevé equations—nonlinear special functions*, Orthogonal Polynomials and Special Functions: Computation and Applications, Lect. Notes in Math. **1883**, Springer, Berlin, 2006, pp. 331–411.

[6] G. Filipuk and W. Van Assche, *Recurrence coefficients of generalized Charlier polynomials and the fifth Painlevé equation*, Proc. Amer. Math. Soc. 141 (2013), 551–562.

[7] G. Freud, *On the coefficients in the recursion formulae of orthogonal polynomials*, Proc. Roy. Irish Acad. Sect. A **76** (1976), 1–6.

[8] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications **35**, Cambridge University Press, 1990.

[9] B. Grammaticos and A. Ramani, *On a novel q-discrete analogue of the Painlevé VI equation*, Phys. Lett. A **257** (1999), 288-292.

[10] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications 98, Cambridge University Press, 2005.

[11] M.E.H. Ismail, *Difference equations and quantized discriminants for q-orthogonal polynomials*, Adv. Appl. Math. **30** (2003), 562–589.

[12] R. Koekoek and R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report no. 98-17 (1998), Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics. Available at http://aw.twi.tudelft.nl/∼koekoek/documents/as98.pdf.

[13] A.P. Magnus, *Freud's equations for orthogonal polynomials as discrete Painlevé equations*, in *Symmetries and Integrability of Difference Equations*, London Math. Soc. Lecture Note Ser. 255, Cambridge University Press, 1999, pp. 228–243.

[14] A. P. Magnus, *Painlevé type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Comput. Appl. Math. **57** (1995), 215–237.

[15] M. A. Olshanetsky and V.-B. K. Rogov, *Modified q-Bessel functions and q-Macdonald functions*, Mat. Sb. **187** 10 (1996) 109–128.

[16] W. Van Assche, *Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials*, in Discrete Equations, Special Functions and Orthogonal Polynomials, World Scientific, 2007, pp. 687–725.